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*ON THE BEHAVIOR OF A CAPILLARY SURFACE IN A WEDGE**

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Abstract.—Estimates above and below are obtained for the height of the equilibrium free surface of a liquid when the liquid partially fills a cylindrical container whose cross section contains a corner with interior angle 2α. The surface is characterized by the condition that its mean curvature be proportional to its height above a reference plane (or, in the case of zero gravity, that the mean curvature be constant), and by the requirement that it meet the container wall with prescribed contact angle γ. It turns out that the qualitative behavior of such a surface near the vertex changes markedly, according as $\alpha + \gamma < \frac{1}{2}\pi$, or $\alpha + \gamma \geq \frac{1}{2}\pi$. In the former case, the surface is either unbounded or fails to exist, while in the latter case every such surface is bounded. Some experimental comparisons are indicated, and an application to the problem of describing the mechanism of water rise in trees is discussed.

The above results describe a limiting case among corresponding properties that hold for surfaces defined over domains with smooth boundaries. This extension is indicated, as well as a formal extension to n-dimensional surfaces; here the interest centers on the fact that it is the mean curvature of an $(n-1)$-dimensional boundary element that controls the local behavior of the n-dimensional solution surface.

1. Consider a volume of liquid that partially fills a cylindrical container Z, forming an equilibrium free surface $\mathcal{S}$, as determined by surface and gravitational forces. The base $\mathfrak{D}$ of Z is to be a right section, bounded in part by two straight segments which meet at a point V and form there an interior angle 2α. We suppose the cylinder to be so oriented that the gravitational field is directed parallel to its generators, and we suppose also that in this configuration $\mathcal{S}$ covers $\mathfrak{D}$ and can be represented by a function $u = u(x,y)$, $(x,y) = \mathbf{x} \in \mathfrak{D}$. We propose to investigate the behavior of $\mathcal{S}$ as V is approached from within the (wedge-shaped) region $\mathfrak{D}$.

The form of $\mathcal{S}$, when such a surface exists, can be determined by the vanishing of the first variation of the sum of the gravitational and surface energies, among all configurations that keep the liquid volume fixed. We are led to the differential equation

$$\operatorname{div}\,(W^{-1}\,\nabla u) = \kappa u + 2H, \quad W = (1 + u_x^2 + u_y^2)^{1/2}, \tag{1a}$$

where H and κ are constants; here κ is twice the reciprocal of the traditional capillary constant, $\kappa = \rho g \sigma^{-1}$, σ being the surface tension on $\mathcal{S}$ and ρ the difference of densities of the liquid and of the gas above the liquid. The sign of the gravitational constant g (and hence of κ) is chosen to be positive when gravity acts to pull the liquid downward toward the base. We do not restrict g to its terrestrial value, and it will be convenient to distinguish the cases $\kappa > 0$, $\kappa = 0$, and $\kappa < 0$.

The boundary condition on $u(\mathbf{x})$ specifies that $\mathcal{S}$ intersect Z with constant angle γ (measured interior to the liquid). We have

$$\cos\gamma = W^{-1}\,\nabla u\cdot\mathbf{n} \tag{1b}$$

on the two straight segments, where $\mathbf{n}$ denotes the outer-directed unit normal. The *contact angle* γ is determined by the physical properties of the three phases at the liquid-solid-gas interface[1]; in practice, it is found experimentally.

If, for a certain choice of materials, $0 \le \gamma < \frac{1}{2}\pi$ (e.g., water and glass), then the fluid is said to "wet" the solid. We need only consider this case, since the "nonwetting" solutions, for which $\frac{1}{2}\pi < \gamma \le \pi$ (e.g., mercury and glass, or water and paraffin), can be obtained from the "wetting" ones by the transformation $\bar\gamma = \pi - \gamma$, $\bar u = -u$, $\bar H = -H$. If $\gamma = \frac{1}{2}\pi$, then (1a, b) admits the trivial solution, $u \equiv$ const. This is the unique solution if (1b) is prescribed on the entire boundary of $\mathfrak{D}$.

Since we shall study the values of $u(x,y) = u(\mathbf{x})$ only at points $\mathbf{x}$ near V, we may suppose $\mathfrak{D}$ to be a circular sector bounded in part by a circular arc centered at V. Denoting the entire boundary of $\mathfrak{D}$ by $\mathcal{C}$, and setting $Tu \equiv W^{-1}\,\nabla u$, we may integrate (1$a$) by parts to obtain[2]

$$\kappa \int_{\mathfrak{D}} u\,d\mathbf{x} + 2H\int_{\mathfrak{D}} d\mathbf{x} = \oint_{\mathcal{C}} Tu\cdot\mathbf{n}\,ds. \tag{2}$$

Thus, with $\mathcal{A}$ the area of $\mathfrak{D}$, $\mathcal{L}$ the length of $\mathcal{C}$, and h the mean height of the liquid,

$$2\,\mathcal{A}\,H = \mathcal{L}\cos\gamma - \kappa\mathcal{A}h. \tag{3}$$

It follows that if $\kappa = 0$, H is determined by $\mathfrak{D}$ and γ. If $\kappa \neq 0$, (3) gives a relationship between H and h.[3]

We remark that the left side of (1a) is twice the mean curvature of $\mathcal{S}$; thus, if $\kappa = 0$, the solutions of (1a) are surfaces of constant mean curvature H. If $\kappa \neq 0$, H can be eliminated by adding a constant to u (which leaves (1b) unchanged); thus, in this case, each solution of (1a) represents a surface whose mean curvature is proportional to its height above some reference plane.

2. The most striking situation is that for which the gravitational field vanishes, so that $\kappa = 0$. Then (1a,b) becomes

$$\operatorname{div}(W^{-1}\,\nabla u) = 2H \qquad \text{in } \mathfrak{D}, \tag{4a}$$

and

$$Tu\cdot\mathbf{n} \equiv W^{-1}\,\mathbf{n}\cdot\nabla u = \cos\gamma \tag{4b}$$

on the straight segments, so that $u(\mathbf{x})$ determines a surface of (positive) constant mean curvature H in $\mathfrak{D}$, subject to the boundary condition (4b). As indicated above, we may suppose $\mathfrak{D}$ to be a circular sector with interior angle 2α at V. We shall show:

A solution[4] *of (4a, b) can exist in* $\mathfrak{D}$ *if and only if* $\alpha + \gamma \ge \frac{1}{2}\pi$.

Thus, for prescribed contact angle γ on the side walls of the wedge, there can be no capillary surface of the indicated form when the interior wedge angle is smaller than $\frac{1}{2}\pi - \gamma$.

The second half of the assertion is a special case of a more general result: *Suppose that* $\alpha + \gamma < \frac{1}{2}\pi$. *Then there is no solution*[4] *of the inequality*

$$\operatorname{div}(W^{-1}\nabla u) \leq 2H < \infty \tag{5a}$$

in $\mathfrak{D}$, *which satisfies*

$$Tu \cdot \mathbf{n} \geq \cos\gamma \tag{5b}$$

on the straight segments.

To demonstrate the result, suppose the existence of such a function $u(x,y)$, with $\alpha + \gamma < \frac{1}{2}\pi$. For any $\tau > \gamma$ satisfying $\alpha + \tau < \frac{1}{2}\pi$, there would be a point O on the bisector of the wedge angle, and segments of length H^{-1} from O to points A and B on the straight segments, such that the angles $V\,O\,A$ and $V\,O\,B$ are each $\frac{1}{2}\pi - (\alpha + \tau)$ (cf. Fig. 1). The circular arc $\widehat{AB}$ about O determines a region Ω bounded by $\widehat{AVB}$, and for τ sufficiently close to$(\frac{1}{2}\pi - \alpha)$, there will hold $\Omega \subset \mathfrak{D}$. For this configuration, we obtain from (5*a*), by an integration by parts,[4]

$$\oint_{\widehat{AVB}} Tu \cdot \mathbf{n}\, ds \leq H \int_\Omega 2\, dx. \tag{6}$$

Choose O as origin of coordinates, and observe that $\operatorname{div} \mathbf{x} \equiv 2$. Thus, (6) can be written

$$\oint_{\widehat{BA}} (Tu - H\mathbf{x}) \cdot \mathbf{n}\, ds + \oint_{\widehat{AVB}} (Tu - H\mathbf{x}) \cdot \mathbf{n}\, ds \equiv I_1 + I_2 \leq 0.$$

On $\widehat{BA}$, $|Tu| < 1$, while $\mathbf{x} \cdot \mathbf{n} \equiv -H^{-1}$. Thus, $I_1 > 0$. On $\widehat{AVB}$, $Tu \cdot \mathbf{n} \geq \cos\gamma$ by (5*b*), while $\mathbf{x} \cdot \mathbf{n} \equiv H^{-1}\cos\tau < H^{-1}\cos\gamma$. Thus also $I_2 > 0$. This contradiction proves the assertion and, as a consequence, the necessity of the previous assertion on the solutions of (4*a*, *b*).

It remains to show that solutions of (4*a*, *b*) do exist if $\alpha + \gamma \geq \frac{1}{2}\pi$. We present such a solution explicitly. When we represent the wedge sides in the form $y \pm x\tan\alpha = 0$, then that part of the lower half of the sphere

$$u^2 + (x - H^{-1}\cos\gamma \csc\alpha)^2 + y^2 = H^{-2}, \tag{7}$$

which lies over $\mathfrak{D}$, is easily seen to satisfy the required conditions.[5]

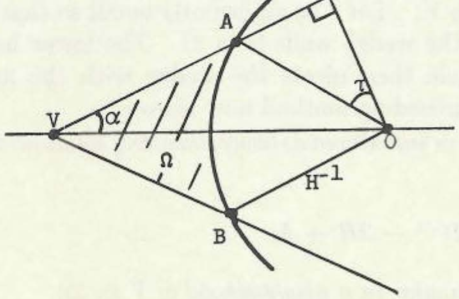


FIG. 1.

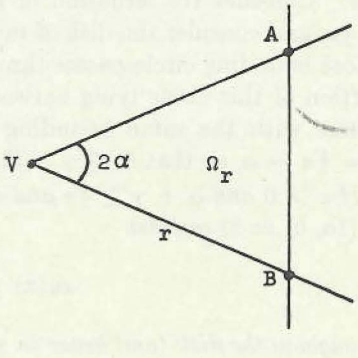


FIG. 2.

3. The question immediately arises as to what actually occurs in a practical situation when the governing equations admit no solution. Such a situation might well be encountered on a space flight should the unwary traveler attempt to sip his apéritif from a glass of square cross section. In this case one could have $\gamma \approx 0$, while $\alpha = \frac{1}{4}\pi$, so that no free surface of the type considered here can exist. It seems to us that a proper clarification of the matter is not to be found in a questioning of the constancy of γ near the corner, but rather in some modification of the class of surfaces admitted to competition in the original variational formulation (cf. §1). The fluid evidently seeks out the corners of the container, to the extent that no finite volume of fluid will yield a simple covering by $\mathcal{S}$ of a domain $\mathfrak{D}$ that extends to the vertex. This view is supported by the material in the following sections.

4. Suppose now that $\kappa > 0$. The methods of section 2 show immediately that if $\alpha + \gamma < \frac{1}{2}\pi$, then there is no solution of (1*a*, *b*) in $\mathfrak{D}$ which is bounded above near V. We may, however, employ a different procedure and strengthen this result. We assert:

If $\kappa > 0$ *and* $\alpha + \gamma < \frac{1}{2}\pi$, *and if* $u(\mathbf{x})$ *satisfies*[4] *(1a, b) in a wedge of interior angle* 2α, *then for all points of distance* r *to the vertex* V *and sufficiently close to* V, *there holds*

$$\kappa u(\mathbf{x}) \leq 2r^{-1} \csc \alpha \cos (\alpha + \gamma) + \kappa r \sin \alpha \sec (\alpha + \gamma) - 2H. \tag{8}$$

The complete demonstration of this result is too lengthy to be presented here. It is based on consideration of a lower hemisphere Σ of radius sufficiently small to permit placing it so that (*a*) its center lies above $\mathcal{S}$ and projects onto the bisector of the vertex angle at V, (*b*) equation (1*b*) is satisfied with contact angle $\bar{\gamma} < \gamma$, and (*c*) no points of Σ project onto the curved part of the boundary of $\mathfrak{D}$. It can then be shown that if Σ is so situated that, in addition, none of its points lies below $\mathcal{S}$ but such that Σ contacts $\mathcal{S}$ at least at one point P, the projection of P onto the plane of $\mathfrak{D}$ will lie interior to the projection of Σ and also interior to $\mathfrak{D}$ (and not on the bounding segments or circle). Once this fact is established, the result follows easily from general properties of solutions of elliptic equations.

5. Consider the situation of the preceding section, except that now $\alpha + \gamma \geq \frac{1}{2}\pi$, and consider the disk of radius δ whose center lies in $\mathfrak{D}$ on the bisector and whose bounding circle passes through V. Let δ be sufficiently small so that the portion of this circle lying between the wedge walls is in $\mathfrak{D}$. The lower hemisphere with the same bounding circle then meets the wedge with the angle $\bar{\gamma} = \frac{1}{2}\pi - \alpha$, so that $\bar{\gamma} \leq \gamma$. The preceding method now shows:

If $\kappa > 0$ *and* $\alpha + \gamma \geq \frac{1}{2}\pi$ *and if* δ *is introduced as above, then any solution*[4] $u(\mathbf{x})$ *of (1a, b) in* $\mathfrak{D}$ *satisfies*

$$\kappa u(\mathbf{x}) \leq 2\delta^{-1} - 2H + \delta\kappa \tag{9}$$

throughout the disk (and hence in particular in a neighborhood of V *in* $\mathfrak{D}$*).*

6. Again consider the same configuration, and let δ be as above. Then

the same method, if now the upper half of the hemisphere is used and placed below the surface $\mathcal{S}$, yields:

If $\kappa > 0$, there holds for any solution $u(\mathbf{x})$ of (1a, b) in $\mathfrak{D}$

$$\kappa u(\mathbf{x}) \geq -2\delta^{-1} - 2H - \delta\kappa \tag{10}$$

throughout the disk.[6]

Note that this last result holds for all values of $\alpha + \gamma$.

7. The preceding material shows that if $\kappa > 0$, then every solution of (1a, b) is bounded below at V. If also $\alpha + \gamma \geq \frac{1}{2}\pi$, then every solution is bounded above. Finally, if $\kappa > 0$ and $\alpha + \gamma < \frac{1}{2}\pi$, then the solutions $u(\mathbf{x})$ satisfy $u(\mathbf{x}) = O(r^{-1})$ at V. Consider now the region $\Omega_r \subset \mathfrak{D}$, bounded by the wedge walls and by the segment $\overline{AB}$ of Figure 2. Then $u(\mathbf{x}) = O(r^{-1})$ implies that the liquid volume $\int_{\Omega_r} u(\mathbf{x})d\mathbf{x} \equiv v(r) = O(r)$ at V. We show that this result cannot be improved, that is, it represents the actual behavior of $v(r)$ for any solution. In fact, a more precise statement can be made:

If $\kappa > 0$ and $\alpha + \gamma < \frac{1}{2}\pi$, then for any solution[4] *$u(\mathbf{x})$ of (1a, b) in $\mathfrak{D}$ there holds*

$$\liminf_{r\to 0}\, r^{-1}\mathcal{V}(r) \geq 2\kappa^{-1}\sqrt{1 - \sin(\alpha - \gamma)}\,\sqrt{1 - \sin(\alpha + \gamma)}.$$

To show this, we need only integrate (1a) by parts in Ω_r and apply (1b). We find $2r\cos\gamma + \oint_{\overline{AB}} Tu \cdot \mathbf{n}\, ds = \kappa \int_{\Omega_r} u(\mathbf{x})d\mathbf{x} + 2H\int_{\Omega_r} d\mathbf{x}$. Since $|Tu| < 1$, there follows $r^{-1}\mathcal{V}(r) > 2\kappa^{-1}[\cos\gamma - \sin\alpha + O(r)]$, which yields the stated result.

8. One consequence of the above is that if $\kappa > 0$, $\alpha + \gamma < \frac{1}{2}\pi$, then $u(\mathbf{x}) >$ const/r at an appreciable set of points as $r \to 0$, thus suggesting that this inequality really holds uniformly for small r. We are presently able to prove such a result when α is not too large (depending on γ), and under an additional hypothesis on the behavior of $u(\mathbf{x})$ at V:

If $\kappa > 0$, $\alpha + \gamma < \frac{1}{2}\pi$, and if $\alpha < \cos\gamma/(\pi - \sin\gamma)$, then for any solution $u(\mathbf{x})$ of (1a, b) which is continuous[7] *in the closure of $\mathfrak{D}$, there holds, letting θ be the polar angle at V measured from the bisecting line,*

$$\kappa u(\mathbf{x}) > r^{-1}\left(1 + \alpha\tan\gamma - \sqrt{\alpha^2\sec^2\gamma - \theta^2}\right) - 2H \tag{11}$$

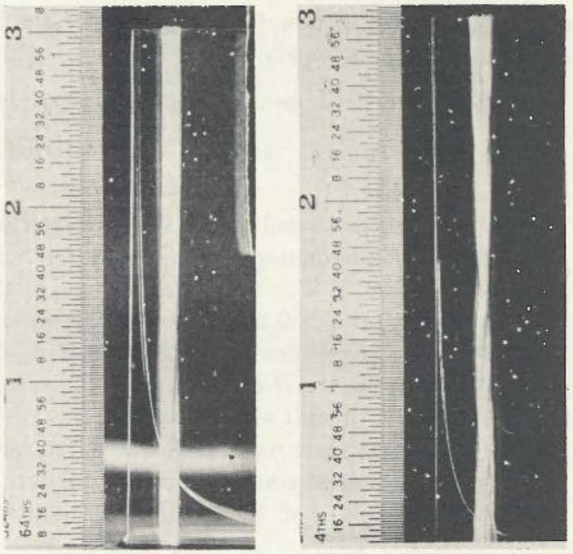
for all sufficiently small r.

This result, together with (8), (9), and (10), characterizes the qualitative behavior of the solutions of (1a, b) in a wedge when $\kappa > 0$. Note that just as in the case $\kappa = 0$, the behavior changes strikingly, depending on whether $\alpha + \gamma < \frac{1}{2}\pi$ or $\alpha + \gamma \geq \frac{1}{2}\pi$.

9. Figure 3 shows the result of a kitchen-sink experiment, in which two microscope slides were held together by a rubber band and separated along one edge by a small piece of plastic so as to form a wedge with interior angle of about 10°. The bottom of the wedge was then dipped into the surface of a reservoir of liquid. In Figure 3a the liquid is water; in (b) it is California olive oil. In both cases, $\alpha + \gamma < \frac{1}{2}\pi$. The accompanying table shows normalized values of the product $ru(r)$ as function of r. ($u(r)$ is the value of $u(\mathbf{x})$ on the wedge wall.) The last value for water in the table was obtained by extrapolation above the slide. The discrepancies from constancy for small r

FIG. 3.

r	(*a*) Water $ru(r)$	(*b*) Oil $ru(r)$
32	51	22
16	46	22
8	43	22
6	41	21
4	40	18
3	41	18
2	≈40	15



Photographs by C. Robertson

(*a*) (*b*)

are probably due to a slight separation of the slides, which was observed along portions of their line of contact.[8]

10. The phenomena described above may conceivably play a role in the mechanism by which water is supplied to the upper leaves of tall trees. As is known, computations based on the rise height of a circular meniscus indicate that the xylem conduits of trees are much too large to raise water to the necessary levels by capillary action alone. However, we have seen that a hollow filament of polygonal cross section will carry liquid to arbitrarily large heights, provided at least one of the interior angles 2α is such that $\alpha + \gamma < \frac{1}{2}\pi$. This appears to be the case for the tracheids of conifers, as well as for some of the conduits occurring in woody angiosperms. Thus, one could expect that some water would be transported through air-filled cells which, according to the generally accepted "cohesion" theory, are considered to have ceased to function in the transport mechanism. Whether the amount of fluid that can be carried in this way suffices to be significant for the water balance of trees is a question that remains to be investigated.[10]

11. Capillary surfaces in small tubes or under small gravitational field are known to remain stable when the gravitational field is reversed in direction. Among others, Reynolds and Satterlee[12] and Concus[13] have investigated the question, and Jetter[14] has reported on experiments in particular cases. The methods we have used in the case $\kappa > 0$ can be used also in this situation ($\kappa < 0$) to characterize the qualitative behavior of the solutions in a wedge. We state here only the results:

(i) *If* $\kappa < 0$ *and* $\alpha + \gamma < \frac{1}{2}\pi$, *and if* $u(\mathbf{x})$ *satisfies*[4] *(1a, b) in a wedge of interior angle* 2α, *then for all points of distance* r *to the vertex* V *and sufficiently close to* V, *there holds* $\kappa u(\mathbf{x}) \leq 2r^{-1} \csc \alpha \cos(\alpha + \gamma) - \kappa r \sin \alpha \sec (\alpha + \gamma) - 2H$.

(ii) *Let* δ *be as in section 5. If* $\kappa < 0$ *and* $\alpha + \gamma \geq \frac{1}{2}\pi$, *then any solution*[4] $u(\mathbf{x})$ *of (1a, b) in* $\mathfrak{D}$ *satisfies* $\kappa u(\mathbf{x}) \leq 2\delta^{-1} - 2H - \delta\kappa$ *throughout the disk (and hence in particular in a neighborhood of* V *in* $\mathfrak{D}$).

(iii) *If* $\kappa < 0$, *then for any solution* $u(\mathbf{x})$ *of (1a, b) in* $\mathfrak{D}$ *there holds* $\kappa u(\mathbf{x}) \geq -2\delta^{-1} - 2H + \delta\kappa$ *throughout the disk.*

(iv) *If* $\kappa < 0$ *and* $\alpha + \gamma < \frac{1}{2}\pi$, *then for any solution* $u(\mathbf{x})$ *of (1a, b) in* $\mathfrak{D}$ *there holds* $\limsup\limits_{r\to 0} r^{-1}\mathcal{V}(r) \leq 2\kappa^{-1}\sqrt{1 - \sin(\alpha - \gamma)}\sqrt{1 - \sin(\alpha + \gamma)}$.

12. This last result may possibly provide an explanation for an unexpected experimental result of Jetter,[14] who found that for $\gamma = \frac{1}{2}\pi$, the capillary columns in sufficiently fine tubes of round and of square cross section were stable when inverted, while for $\gamma = 0$ the round column was stable and the square one unstable on inversion. The result (iv) above shows that in the case of a square with $\gamma = 0$ and $\kappa < 0$, there must be a sequence of points tending to the corners, along which $u(\mathbf{x}) \rightarrow -\infty$. This suggests that the fluid surface would extend to the bottom of a container when it is inverted, providing a path for air to reach the bottom (the top after inversion) of the fluid mass, thus relieving the pressure differential which would be necessary to prevent the column from sliding as a rigid mass down the tube.

13. In this paper we have restricted our attention to a particular and simple configuration in order to emphasize what seem to us the most striking features of our results. In a larger work which is now in preparation, we shall present complete demonstrations of the results we have stated here; we shall obtain them as limiting cases of more general results, which apply to boundary arcs having a continuously turning tangent, and as special cases of properties of solutions $u = u(x_1,\cdots,x_n)$ of the equation $\sum_{i=1}^{n} (W^{-1}u_{x_i})_{x_i} = \kappa u + nH$ defined over an n-dimensional region $\mathfrak{D}$, and satisfying $W^{-1}\mathbf{n}\cdot\nabla u = \cos\gamma$, where γ is subject to certain restrictions, on the bounding surface Σ of $\mathfrak{D}$.

A typical result is the following:

Let $u = u(x_1, \cdots, x_n)$ *be a solution of the inequality* $\sum_{i=1}^{n} (W^{-1}u_{x_i})_{x_i} \leq nH < \infty$ *in a one-sided neighborhood of a point* P *of an* $(n-1)$*-dimensional surface* Σ *with mean curvature* H_Σ. *Suppose that* $H_\Sigma(P) > nH/(n-1)$. *Then* $\cos\gamma < 1$ *at* P.

The example of the hypersphere of radius H^{-1} shows that one can have $\cos\gamma \equiv 1$ on a surface Σ for which $H_\Sigma \equiv H$. A less trivial example shows that the assertion is in general false if $H_\Sigma(P) \leq nH/(n-1)$. Finally, for any prescribed $H_\Sigma(P)$ there are surfaces Σ, and corresponding solutions $u(\mathbf{x})$ in a one-sided neighborhood N_Σ of Σ at P, that yield arbitrarily prescribed values $\cos\gamma \equiv \text{const} < 1$ on Σ (although the size of N_Σ must then be restricted, depending on γ).

Observe that in the case $n > 2$ it is the mean curvature of Σ that is crucial in determining the local qualitative behavior (near Σ) of the solutions. This fact would not be apparent in a discussion limited to two dimensional surfaces.

We should like to thank A. Acrivos, C. Concus, D. Kilbridge, and P. M. Ray for helpful comments.

Note added in proof: After submitting this paper we obtained the explicit form of the most general solution for the case $\alpha + \gamma < \frac{1}{2}\pi$, to within an additive constant. The resulting improvements in estimates (8), (11), and (i) of § 11 will be given elsewhere.[15]

*This investigation was performed in part under auspices of the U. S. Atomic Energy Commission and in part under contract AF 49(638)1345.

[1] Adam, Neil K., *The Physics and Chemistry of Surfaces* (London: Oxford Univ. Press, 1941).

[2] We consider in this paper only those solutions $u(x,y)$ which are continuously differentiable up to and on the (open at V) lateral sides of the wedge. However, except for the result of §8, no hypothesis is made on the behavior of the solutions at the vertex V. They need not be defined at V, nor is any bound imposed on them as V is approached. The validity of (2) under these hypotheses follows from the fact that $|Tu| < 1$ for any function $u(x,y)$.

[3] Note that ($1a, b$) do not suffice to determine this relationship, since γ is not supposed known on the curved part of C. However, the specific values of H and h do not qualitatively affect the results we shall present.

[4] Cf. ref. 2. We emphasize again that nore striction need be imposed on the behavior of $u(\mathbf{x})$ as V is approached.

[5] The reader may wish to verify that this solution satisfies the boundary condition ($4b$) over the *entire* perimeter (except, of course, for the vertices) of any polygon Π whose (extended) sides circumscribe the circle centered at $(H^{-1}\cos\gamma \csc\alpha, 0)$ and inscribed in the wedge whenever Π lies in the closed disk $(x - H^{-1}\cos\gamma\csc\alpha)^2 + y^2 \leq H^{-2}$. If Π contains a vertex exterior to this disk, no solution of ($4a, b$) in Π can exist.

[6] If ($1b$) is prescribed on all of $\mathfrak{C}$ (the vertices excepted), then there holds $u(x) > -2H/\kappa$ in $\mathfrak{D}$; in fact, one could obtain by the same method a still stronger estimate, depending on the geometry of $\mathfrak{D}$.

[7] In the topology of the compactified number line. Under this hypothesis, the results of §7 imply $u(\mathbf{x}) \to \infty$ as $\mathbf{x} \to V$.

[8] For small α, the form of these curves can be inferred from heuristic consideration (cf. ref. 1, pp. 371 and 372). See also ref. 9, pp. 25 and 26.

[9] Boys, C. V., *Soap Bubbles, Their Colours and the Forces which Mould Them* (London: Society for Promoting Christian Knowledge, 1924).

[10] We thank Eugene Isaacson for bringing to our attention this possible application of our ideas. We remark that the view that water is transported along the surface walls of cells, in contradistinction to the cohesion theory, which requires that cells be filled with liquid, has found one of its stronger proponents in Pfeffer (ref. 11, p. 202 ff.).

[11] Pfeffer, W., *Pflanzenphysiologie* (Leipzig: Verlag Wilh. Engelmann, 1897), 2nd ed., vol. 1.

[12] Reynolds, W. C., and H. M. Satterlee, "Liquid propellant behavior at low and zero g," in *The Dynamic Behavior of Liquids in Moving Containers*, ed. H. N. Abramson (NASA SP-106) (Washington, D. C.: Government Printing Office, 1966), pp. 387–439.

[13] Concus, Paul, "Static menisci in a vertical right circular cylinder," *J. Fluid Mech.*, **34**, 481–495 (1968).

[14] Jetter, Robert I., "Orientation of fluid surfaces in zero gravity through surface tension effects," in *Advances in the Astronautical Sciences*, ed. E. T. Benedict and R. W. Halliburton (Baltimore: American Astronautical Society, 1963), vol. 14, pp. 60–71.

[15] Concus, P., and R. Finn, "On a class of capillary surfaces," *J. Analyse Math.*, to appear.